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Dirac Equation and the Ivanenko–Landau–Kähler Equation

Yu. N. Obukhov[1] and S. N. Solodukhin[1,2]

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We consider spinor theory within the framework of an inhomogeneous differential forms formalism. We also consider the possibility of describing fermions with the Ivanenko–Landau–Kähler equation. The relations between these two equations are studied.

1. INTRODUCTION

The Dirac equation (Bade and Jehle, 1953) presents the well-known standard description of fermions, the half-integer-spin particles. Spinors which realize the double-valued representations of the orthogonal space-time symmetry group were introduced in the mathematical literature in 1913 by Cartan (1913) and they later were rediscovered by Dirac (1928) in an attempt to understand the behavior of electrons in a magnetic field. The Dirac spinors are widely used in quantum field theory; however, these do not have a simple interpretation as geometrical objects. In particular, this fact leads to ambiguities in the definition of fermion fields on arbitrary topologically nontrivial manifolds: there exist nonequivalent recipes for introducing the spinor structure on curved space-time (Budinich and Trautman, 1988). In addition, it was noticed long ago (see, e.g., Zhelnorovich, 1982) that the Dirac equation is in fact equivalent to a system of nonlinear tensor equations. In the present paper we address another approach, and construct the spinor theory in terms of antisymmetric tensor fields (differential forms) and linear field equations. Such a reformulation of the theory of fermions within the geometrical framework of exterior

[1]Department of Theoretical Physics, Moscow State University, Moscow 117234, Russia.
[2]International Centre for Theoretical Physics, Trieste, Italy.

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forms is of particular interest for the description of half-integer-spin fields on curved manifolds.

It is less known that fermions can be described by inhomogeneous differential forms which satisfy the Ivanenko–Landau–Kähler (ILK) equation. The latter has a long and dramatic history (see, e.g., Ivanenko and Landau, 1928; Kahler, 1962; Graf, 1978; Becher and Joos, 1983; Benn and Tucker, 1983; Ivanenko *et al.*, 1985; Ivanenko and Obukhov, 1985; Leonovich, 1983; Plebanski, 1984; Budinich and Bugajska, 1985; and references therein). The ILK equation has not yet found wide application; however, it is worth mentioning its use in lattice models (Graf, 1978; Göckeler and Joos, 1984) and in the description of generations (Banks *et al.*, 1982). Of the most recent developments we would like to mention the ILK-based string models (Solodukhin, 1989, 1991) and its to some extent unexpected role in Witten's (1988) topological field theory, where the ILK equation describes the ghost sector of the model.

Both the Dirac and ILK equations can be formulated in terms of exterior forms, so it seems interesting to investigate their possible relations. The point is central in the present paper.

The paper is organized as follows. Section 2 contains a brief review of the ILK theory; several representations of the ILK equation are discussed.

In Section 3 we consider the theory of algebraic spinors within the framework of representations of the Clifford algebras. The spinor is defined as an element of the left minimal ideal of the Clifford algebra.

Section 4 is devoted to the discussion of a somewhat different formulation of spinors which is equivalent, in flat Minkowski space-time, to the algebraic formulation. The problem of equivalence in Riemannian space-time is more complicated. The reduction of the Dirac equation in R^4 to the surface $R^2 \subset R^4$ is considered, showing the emergence of the ILK equation on the latter.

The conjugation operations are studied in Section 5. We demonstrate that reduction of the Dirac equation not only induces on a two-surface the ILK equation, but also the conjugation law on it.

In Section 6 we treat the ILK equation in M^4 as the result of reduction of the usual Dirac equation from the 8-dimensional space M^8. It is shown that the conjugation of the ILK field as well as the internal (right) symmetry group are determined by the signature of the M^8 metric. The anomalous signature yields the noncompact symmetry group, which leads to complications with quantization of the ILK theory.

Finally, Section 7 contains general discussion and the summary of the results obtained.

2. THE IVANENKO–LANDAU–KÄHLER EQUATION

Let M^4 be the Minkowski space-time with the metric tensor $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. Elements of the exterior form algebra $\Lambda^*(M^4) = \bigoplus_{p=0}^{4} \Lambda^p(M^4)$ are the nonhomogeneous exterior forms

$$\phi = \sum_{p=0}^{4} \frac{1}{p!} \varphi_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \tag{2.1}$$

where $\{dx^\mu\}$ is the *I*-form basis of the cotangent space $T^*(M^4)$, dual to the coordinate basis $\{\partial_\mu\}$. In $\Lambda^*(M^4)$ two differential operators are defined: exterior differential d and codifferential δ. For the p-form

$$\psi = \frac{1}{p!} \psi_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

$$d\psi = \frac{1}{p!} \partial_{\mu_1} \psi_{\mu_2 \cdots \mu_{p+1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}}$$

$$\delta\psi = -\frac{1}{(p-1)!} \partial^\alpha \psi_{\alpha\mu_1 \cdots \mu_{p-1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p-1}}$$

The codifferential δ is adjoint to d with respect to the natural scalar product of p-forms,

$$(\omega, \psi) = \int_{M^4} \frac{1}{p!} \omega_{\mu_1 \cdots \mu_p} \psi^{\mu_1 \cdots \mu_p} d^4x$$

and it can be represented in the form $\delta = *^{-1} d*$, where $*\colon \Lambda^p \to \Lambda^{4-p}$ is the Hodge dualization operator. The evident properties are as follows

$$\begin{gathered} d^2 = 0, \qquad \delta^2 = 0 \\ -(d\delta + \delta d) = \Box = \partial_\mu \partial^\mu \end{gathered} \tag{2.2}$$

In 1928, independently of Dirac (1928), the relativistic wave equation was proposed by Ivanenko and Landau (1928) for the fields (2.1) which correctly describe the behavior of an electron in an external magnetic field. This equation,

$$\{i(d - \delta) - m\}\phi = 0 \tag{2.3}$$

was rediscovered in 1960 by the mathematician Kähler (1962) and is often called the Dirac–Kähler equation.

With the help of (2.2) one can easily see that, like the standard Dirac equation, equation (2.3) is the "square root" of the Klein–Gordon equation,

$$(\Box + m^2)\phi = 0$$

This similarity suggests deeper relations between the two equations. We shall study this aspect within the framework of the theory of the Clifford algebra representations.

First, let us note that there exists a natural correspondence between exterior and Clifford algebras. Namely, one can explicitly construct the complex Clifford algebra $C_{1,3}(M^4)$ on the Minkowski spacetime with the metric signature $(1, 3)$ by introducing the new algebraic operation in $\Lambda^*(M^4)$: the Clifford product of exterior forms. For the basis *1*-forms dx^μ the latter is defined by the formula

$$dx^\mu \vee dx^\nu \equiv dx^\mu \wedge dx^\nu + \eta^{\mu\nu} \tag{2.4}$$

Then an arbitrary complex nonhomogeneous form (2.1) may be expressed as an element of the Clifford algebra

$$\phi = \sum_{p=0}^{4} \frac{1}{p!} \varphi_{\mu_1 \cdots \mu_p} dx^{\mu_1} \vee \cdots \vee dx^{\mu_p} \tag{2.5}$$

The algebra $C_{1,3}(M^4)$ is defined as the formal algebra with respect to the Clifford product $\vee$ which is spanned by the basis elements,

$$\{1, dx^\mu, \ldots, dx^{\mu_1} \vee \cdots \vee dx^{\mu_p}, \ldots, dx^0 \vee \cdots \vee dx^3\} \tag{2.6}$$

constructed from the generating elements dx^μ which in view of (2.4) satisfy the anticommutation relation

$$dx^\mu \vee dx^\nu + dx^\nu \vee dx^\mu = 2\eta^{\mu\nu} 1. \tag{2.7}$$

Clearly, the complex dimension of $C_{1,3}(M^4)$ coincides with that of $\Lambda^*(M^4)$ and is equal to $2^4 = 16$.

With the help of (2.4) the ILK equation (2.3) is rewritten as

$$(i\, dx^\mu \vee \partial_\mu - m)\phi = 0 \tag{2.8}$$

Equations (2.7) and (2.8) suggest the correspondence between dx^μ and the Dirac matrices γ_μ, and between the Clifford product $\vee$ and the usual matrix product.

Indeed, let us consider a complex 4×4 matrix ψ (Becher and Joos, 1983; Ivanenko and Obukhov, 1986; Ivanenko *et al.*, 1985)

$$\psi = \sum_{p=0}^{4} \frac{1}{p!} \varphi_{\mu_1 \cdots \mu_p} \Gamma^{\mu_1 \cdots \mu_p} \tag{2.9}$$

The spin tensors

$$\Gamma_{\mu_1 \cdots \mu_p} = \gamma_{[\mu_1} \cdots \gamma_{\mu_p]}, \qquad p = 0, 1, \ldots, 4$$

together with the unit matrix realize the basis of the 4-dimensional Dirac algebra, defined by the relation

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} I$$

The inverse transformation is

$$\varphi_{\mu_1 \cdots \mu_p} = \frac{1}{4}(-1)^{p(p-1)/2} \operatorname{Tr}(\psi \Gamma_{\mu_1 \cdots \mu_p}) \tag{2.10}$$

For complex conjugates,

$$\bar{\varphi}_{\mu_1 \cdots \mu_p} = \frac{1}{4} \operatorname{Tr}(\bar{\psi} \Gamma_{\mu_1 \cdots \mu_p}) \tag{2.11}$$

where $\bar{\psi} = \gamma_0 \psi^+ \gamma_0$, and we have used $\gamma_0 \gamma_\mu^+ \gamma_0 = \gamma_\mu$. When the tensors $\varphi_{\mu_1 \cdots \mu_p}$ satisfy (2.3) one obtains the equation for the matrix field ψ (Becher and Joos, 1983; Ivanenko and Obukhov, 1985; Ivanenko *et al.*, 1985)

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{2.12}$$

which resembles the Dirac equation, but unlike the latter, ψ is not a column but a 4×4 matrix.

Under the Lorentz transformations

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$\varphi^{\mu_1 \cdots \mu_p} \to \varphi'^{\mu_1 \cdots \mu_p} = \Lambda^{\mu_1}{}_{\nu_1} \cdots \Lambda^{\mu_p}{}_{\nu_p} \varphi^{\nu_1 \cdots \nu_p}$$

the field (2.9) transforms as follows:

$$\psi \to \psi^1 = S\psi S^{-1} \tag{2.13}$$

where the matrix S realizes the spinor representation of the Lorentz group

$$S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu$$

The Lorentz-invariant Lagrangian for the field ψ has the form

$$L = \operatorname{Tr}\left\{\frac{i}{2}(\partial_\mu \bar{\psi}\gamma^\mu \psi - \bar{\psi}\gamma^\mu \partial_\mu \psi) + m\bar{\psi}\psi\right\} \tag{2.14}$$

Let us mention here that quantization of the ILK theory requires the indefinite metric in the Hilbert space, and this leads to some difficulties. Their source is the invariance of the theory under the noncompact internal ("right") symmetry group $SU(2, 2)$ (Benn and Tucker, 1983; Ivanenko and Obukhov, 1985; Ivanenko *et al.*, 1985), evident from (2.14). The type of quantization is then an open problem; both the Fermi–Dirac (Benn and Tucker, 1983; Ivanenko and Obukhov, 1985; Ivanenko *et al.*, 1985) and the Bose–Einstein (Leonovich, 1983; Plebanski, 1984; Satikov and Stragev, 1987) approaches have been developed.

The ILK equation has several useful representations (Ivanenko *et al.*, 1985). Besides the ones mentioned above, let us describe one more. Let us introduce a 16-component column $\psi_A, A = 1, \ldots, 16$, the elements of which are ψ_{ij} and the index A is understood as the pair (i, j). Define the 16×16 matrices Γ_μ which satisfy

$$\Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\eta_{\mu\nu} I$$

by the expression

$$(\Gamma_\mu)_{AA'} = (\gamma_\mu)_{ii'} \delta_{jj'}$$

where $A = (i, j)$, $A' = (i', j')$.

Then the ILK equation (2.3), (2.12) is transformed to the form

$$i\Gamma^\mu_{AB} \partial_\mu \psi_B - m\psi_A = 0 \tag{2.15}$$

3. THE CLIFFORD ALGEBRA AND ALGEBRAIC SPINORS

Let us consider now the theory of spinors in M^4 within the framework of the Clifford algebra representations. It will be more convenient to work with the real Clifford algebras.

Let N^4 be the 4-dimensional space with the metric $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. We will assume this is different from the Minkowski spacetime M^4. Denote the orthonormal basis in N^4 by elements $\{e_\mu, \mu = 0, 1, 2, 3\}$. The real Clifford algebra $R_{1,3}(N^4)$ of the space N^4 is defined as the algebra spanned on

$$\{1, e_\mu, \ldots, e_{\mu_1}, \ldots, e_{\mu_p}, \ldots, e_0 e_1 e_2 e_3\} \tag{3.1}$$

$\mu_1 < \mu_2 < \cdots < \mu_p, p = 0, 1, \ldots, 4$, which are constructed with the help of the algebraic operation defined on the basis as

$$e_\mu e_\nu + e_\nu e_\mu = 2\eta_{\mu\nu} \tag{3.2}$$

An element $Y \in R_{1,3}(N^4)$ has the form

$$Y = \sum_{p=0}^{4} \frac{1}{p!} y^{\mu_1 \cdots \mu_p} e_{\mu_1} \ldots e_{\mu_p} \tag{3.3}$$

where $y^{\mu_1 \cdots \mu_p}$ are completely antisymmetric real tensors.

Now let N^5 be the real 5-dimensional vector space with the metric $\hat{\eta}_{\alpha\beta} = (+1, -1, -1, -1, +1)$ and let $\hat{e}_\alpha, \alpha = 0, 1, \ldots, 4$, denote an orthonormal basis in N^5. We embed N^5 in such a way that the basis e_μ coincides with the first four vectors of $\hat{e}_\alpha$. The real Clifford algebra $R_{2,3}(N^5)$, related to N^5, is easily seen to be isomorphic to the direct sum $R_{2,3} \approx R_{1,3} \oplus R_{1,3}$.

Indeed, let us denote

$$i = -\hat{e}_0\hat{e}_1\hat{e}_2\hat{e}_3\hat{e}_4 = -e_0e_1e_2e_3\hat{e}_4 \tag{3.4}$$

Clearly, i belongs to the center of $R_{2,3}$, i.e., $i\hat{e}_\alpha = \hat{e}_\alpha i$. One checks at once that $i^2 = -1$. Hence, the maximal element (3.4) plays in $R_{2,3}$ the role of the imaginary unit.

An element $\hat{Y} \in R_{2,3}(N^5)$ can be represented in the form

$$\hat{Y} = \sum_{p=0}^{5} \frac{1}{p!} \hat{y}^{\alpha_1 \cdots \alpha_p} \hat{e}_{\alpha_1} \ldots \hat{e}_{\alpha_p} = Y + i\psi \tag{3.5}$$

where Y is given by (3.3) with $y^{\mu_1 \cdots \mu_p} = \hat{y}^{\mu_1 \cdots \mu_p}$ and

$$\Psi = \sum_{p=0}^{4} \frac{1}{p!} \psi^{\mu_1 \cdots \mu_p} e_{\mu_1} \ldots e_{\mu_p}$$

where

$$\psi^{\mu_1 \cdots \mu_p} = \frac{(-1)^{p(p-1)/2}}{(5-p)!} \hat{\varepsilon}^{\mu_1 \cdots \mu_p}{}_{\alpha_1 \cdots \alpha_{5-p}} \hat{y}^{\alpha_1 \cdots \alpha_{5-p}}$$

The Levi–Civita symbol in N^5 is such that $\hat{\varepsilon}^{01234} = +1$. Thus an element of $R_{2,3}(N^5)$ can be represented as the formal sum (3.5) in which the "imaginary" and "real" parts Y, ψ are the elements of $R_{1,3}(N^4)$. In other words, we have shown that $R_{2,3}(N^5) \cong C_{1,3}(N^4)$.

Let A be the Clifford algebra. A subset $S \subset A$ is called a left ideal if $as \in S$ holds for any $a \in A, s \in S$. The left ideal is called minimal when it does not contain nontrivial left ideals. One can obtain a left ideal with the help of the idempotent element $P \in A$, such that $P^2 = P$. Then $S = AP$. One calls two idempotents "orthonormal" if $PP' = P'P = 0$. Finally, an idempotent is called primitive if it is impossible to decompose it into the sum of two orthonormal idempotents. To any primitive idempotent there corresponds a minimal left ideal. In the Clifford algebra one can construct the complete set of primitive idempotents $\{P_i\}$ such that

$$\begin{aligned} &P_iP_j = 0, \qquad i \neq j \\ &P_iP_i = P_i \qquad \text{(no sum)} \\ &\sum_i P_i = 1 \end{aligned} \tag{3.6}$$

Then A may be decomposed into the sum of minimal left ideals

$$A = \sum_i S_i, \qquad S_i = AP_i \tag{3.7}$$

The left multiplication evidently defines a linear representation of the Clifford algebra A on minimal left ideals,

$$A \times S_i \to S_i$$
$$a \in A,\ s \in S_i \to as \in S_i$$

By construction these representations are irreducible.

Notice that with the help of an arbitrary inversible element $u \in A$ one can obtain from $\{P_i\}$ another complete set of primitive idempotents $P'_i = uP_iu^{-1}$ which satisfy (3.6). Moreover, there always exists an element u_{ij} such that $P_i = u_{ij}P_ju_{ij}^{-1}$. Hence all the representations of the Clifford algebra on the minimal left ideals are equivalent.

One can prove (Lounesto, 1986) that in a real algebra $R_{p,q}$ there are $k = q - \tau(q-p)$ mutually commuting elements $\lambda_1, \ldots, \lambda_k$ with unit square $\lambda_1^2 = \cdots = \lambda_k^2 = 1$. Here $\tau(n)$ is the Radon–Hurwitz number, which for any $n \in Z$ is determined by the recursion formula $\tau(n+8) = \tau(n) + 4$, and $\tau(0) = 0$, $\tau(1) = 1$, $\tau(2) = \tau(3) = 2$, $\tau(4) = \cdots = \tau(7) = 3$. Clearly

$$P_a^{\varepsilon_a} = \frac{1}{2}(1 + \varepsilon_a\lambda_a), \qquad \varepsilon_a = \pm 1$$

are idempotents, and

$$P_i = P_1^{\varepsilon_i} \cdot P_2^{\varepsilon_2}, \ldots, P_k^{\varepsilon_k} \tag{3.8}$$

are primitive idempotents. The total number of these is equal to 2^k—the number of different combinations of sign coefficients $\{\varepsilon_1, \ldots, \varepsilon_k\}$. For the case under consideration, $R_{2,3}(N^5)$, $k = 2$ and hence there exist four primitive idempotents.

As two independent commuting elements we can choose $\lambda_1 = ie_3$, $\lambda_2 = ie_1e_2$. Clearly $\lambda_1\lambda_2 = \lambda_2\lambda_1$, $\lambda_1^2 = \lambda_2^2 = 1$. Then the complete set (3.8) is as follows:

$$\begin{aligned} P_1 &= \frac{1}{4}(1 + ie_3)(1 + ie_1e_2), \qquad & P_2 &= \frac{1}{4}(1 - ie_3)(1 + ie_1e_2) \\ P_3 &= \frac{1}{4}(1 + ie_3)(1 - ie_1e_2), \qquad & P_4 &= \frac{1}{4}(1 - ie_3)(1 - ie_1e_2) \end{aligned} \tag{3.9}$$

Let us describe the minimal left ideals which correspond to (3.9). Consider the action of P_i, $i = 1, \ldots, 4$, from the right on an arbitrary element $\hat{Y} \in R_{2,3}(N^5)$. Taking the latter in the form (3.5), one gets

$$\begin{aligned} Y_{(i)} &= \hat{Y}P_i \in S_i = R_{2,3}P_i \\ Y_{(i)} &= [\psi^1_{(i)} + \psi^2_{(i)}e_0e_1 + \psi^3_{(i)}e_0 + \psi^4_{(i)}e_1]P_i \end{aligned} \tag{3.10}$$

where $\psi^j_{(i)}$ are "complex" coefficients constructed from the components of (3.5).

In the ith minimal ideal the elements

$$1P_i,\ e_0e_1P_i,\ e_0P_i,\ e_1P_i \tag{3.11}$$

can be considered as the basis of the four-dimensional complex linear space S_i. Correspondingly, the quantities $\psi^j_{(i)}$ are naturally interpreted as the components of elements $Y_{(i)}$ in S_i with respect to the basis (3.11). We call any of equivalent ideals S_i the spinor space S. Its complex dimension is equal to $2^{4/2}=4$. The above construction is well known as "algebraic spinors." Equations (3.10), (3.11) show that S is isomorphic to the complex Clifford algebra $C_{1,1}$ of a two-dimensional space (spanned by the vectors e_0, e_1). This fact is a particular case of the general theory of the Clifford algebra representations where the spinor representations of C_{2m} are realized on C_m (Rashevsky, 1955; Chevalley, 1954). We will use this below.

The components of an element of spinor space (of spinor) may be conveniently arranged on a column

$$\psi_{(i)} = \begin{pmatrix} \psi^1_{(i)} \\ \psi^2_{(i)} \\ \psi^3_{(i)} \\ \psi^4_{(i)} \end{pmatrix} \tag{3.12}$$

Here the index (i) denotes the minimal left ideals to which the given spinor belongs.

Acting from the left by the elements $\hat{e}_\alpha$ on the basis (3.11), one finds that the vectors $\hat{e}_\mu$, $\mu = 0, 1, 2, 3$, and $\hat{e}_4$ are represented by 4×4 matrices

$$\hat{e}_0\colon\quad \hat{E}_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{3.13a}$$

$$\hat{e}_1\colon\quad \hat{E}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{3.13b}$$

$$\hat{e}_2\colon\quad \hat{E}_2 = \pm \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{3.13c}$$

$$\hat{e}_3: \quad \hat{E}_3 = \pm \begin{bmatrix} \begin{matrix} -i & 0 \\ 0 & -i \end{matrix} & 0 \\ 0 & \begin{matrix} i & 0 \\ 0 & i \end{matrix} \end{bmatrix} \tag{3.13d}$$

$$\hat{e}_4: \quad \hat{E}_4 = i\hat{E}_5 = i\hat{E}_0\hat{E}_1\hat{E}_2\hat{E}_3 \tag{3.13e}$$

The signs in (3.13c) and (3.13d) depend on the choice of the minimal left ideals: in (3.13c) for S_1 and S_2 one must take $(+)$, while for S_3 and S_4 $(-)$; in (3.13d) $(+)$ is for S_1, S_3 and $(-)$ for S_2, S_4.

The matrices (3.13) satisfy

$$\hat{E}_\alpha\hat{E}_\beta + \hat{E}_\beta\hat{E}_\alpha = 2\hat{\eta}_{\alpha\beta} \tag{3.14}$$

and these are a possible set of Dirac matrices.

Thus the action of an element of the Clifford algebra (3.5) on spinors (3.12) in the given basis (3.11) is represented simply as the matrix multiplication from the left by the 4×4 matrix Y which is determined from (3.5) by substituting $\hat{e}_\alpha$ by $\hat{E}_\alpha$, (3.13). Hence the representation of the algebra $R_{1,3}(N^4) \subset R_{2,3}(N^5)$ [put $\psi = 0$ in (3.5)] is defined on S_i.

This also gives the spinor representation of the orthogonal group on S_i. To show this let us consider the subset of invertible elements Y in $R_{1,3}$. These define an internal automorphism of the Clifford algebra

$$R_{1,3} \to YR_{1,3}Y^{-1}$$

By construction, the Minkowski space N^4 is embedded in $R_{1,3}(N^4)$ as the set of elements of the form $X = x^\mu e_\mu$. The Clifford group $G_c(N^4)$ of the space N^4 is the set of invertible elements in $R_{1,3}$ such that the relevant automorphisms leave $N^4 \subset R_{1,3}(N^4)$ invariant, i.e.,

$$G_c(N^4) = \{Y \in R_{1,3}(N^4) \,|\, YXY^{-1} \in N^4, \forall X = \kappa^\mu e_\mu \in N^4\}$$

In particular, for $Y \in G_c(N^4)$, one finds

$$Ye_\mu Y^{-1} = \Lambda_\mu{}^\nu e_\nu \tag{3.15}$$

Squaring (3.15), we get

$$Ye_\mu e_\nu Y^{-1} = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta e_\alpha e_\beta$$

and for the symmetric part

$$\eta_{\mu\nu} = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta \eta_{\alpha\beta}$$

Thus, $\Lambda_\mu{}^\nu \in O(1, 3)$, and (3.15) describes the homomorphisms of the Clifford group in the orthogonal group. This map is not one-to-one, since for any element aY, $a \in R$, in (3.15) there corresponds the same $\Lambda_\mu{}^\nu$. Hence one uses the normalization condition.

In $R_{1,3}(N^4)$ involutions are defined: the main involution α which maps (3.3) into

$$\alpha(Y) = \sum_{p=0}^{4} \frac{1}{p!}(-1)^p \eta^{\mu_1 \cdots \mu_p} e_{\mu_1} \ldots e_{\mu_p} \tag{3.16}$$

and the main anti-involution β,

$$\beta(Y) = \sum_{p=0}^{4} \frac{1}{p!}(-1)^{p(p-1)/2} \eta^{\mu_1 \cdots \mu_p} e_{\mu_1} \ldots e_{\mu_p} \tag{3.17}$$

with the properties

$$\alpha^2 = \beta^2 = id, \qquad \alpha(Y^{-1}) = (\alpha(Y))^{-1}$$

$$\alpha(ab) = \alpha(a)\alpha(b), \qquad \beta(Y^{-1}) = (\beta(Y))^{-1}$$

$$\beta(ab) = \beta(b)\beta(a), \qquad a, b \in R_{1,3}$$

Applying β to (3.15), one gets

$$\beta(Y)^{-1} e_\mu \beta(Y) = \Lambda_\mu{}^\nu \beta(e_\nu)$$

Subtracting (3.15) from this, we obtain

$$\beta(Y) Y e_\mu = e_\mu \beta(Y) Y$$

Thus $\beta(Y)Y$ is a scalar.

The set

$$\mathrm{Pin}(N^4) = \{ Y \in G_c(N^4) \,|\, \beta(Y)Y = 1 \} \tag{3.18}$$

is called the spinor group. Equation (3.15) gives the homomorphism

$$\mathrm{Pin}(N^4) \to O(1, 3)$$

This map is two-to-one: any rotation $\Lambda_\mu{}^\nu$ corresponds to the two elements $\pm Y$ of the spinor group. The subset of even elements in $\mathrm{Pin}(N^4)$ forms the subgroup which is called the special spinor group,

$$\mathrm{Spin}(N^4) = \{ Y \in \mathrm{Pin}(N^4) \,|\, \alpha(Y) = Y \}$$

In this case $\det \Lambda_\mu{}^\nu = \pm 1$, and (3.15) gives the homomorphism

$$\mathrm{Spin}(N^4) \to SO(1, 3)$$

Thus, we obtain finally the representation of the Lorentz group on the spinor space which is related via the elements of $\mathrm{Spin}(N^4) \subset R_{1,3}(N^4)$. For example, the infinitesimal rotation

$$\Lambda_\mu{}^\nu = \delta_\mu{}^\nu + \omega_\mu{}^\nu, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

yields via (3.15)

$$Y = 1 + \frac{1}{4}\,\omega_{\mu\nu} e^{\mu} e^{\nu}$$

The relevant matrix realization on the spinor space is given by the 4×4 matrix

$$Y = 1 + \frac{1}{4}\,\omega_{\mu\nu} \hat{E}^{\mu} \hat{E}^{\nu}$$

where $\hat{E}^{\mu}$ is as in (3.13).

Returning now to the ILK equation (2.3), (2.8), we note that the field (2.5) gives a concrete realization of (3.5), while the abstract space N^4 can be idempotent identified with M^4, and e^{μ} with dx^{μ}. Then decomposing ϕ, (2.5), into the sum of four spinor fields $\phi_{(i)} = \phi P_i$ and substituting into (2.8), we obtain four independent spinor equations

$$(i\, dx^{\mu} \vee \partial_{\mu} - m)\phi_{(i)} = 0, \qquad i = 1, \ldots, 4 \tag{3.19}$$

In the matrix form (3.19) in the basis (3.11) one gets

$$(i\hat{E}^{\mu}\partial_{\mu} - m)\psi_{(i)} = 0 \tag{3.20}$$

that is, the ILK equation is reduced to the four independent Dirac equations for the spinor belonging to different minimal left ideals. Formally, this was apparent already in (2.12).

We conclude the section with the remark that in general Φ might not be decomposable into elementary spinors, since arbitrary spacetimes do not admit covariantly constant idempotents (Graf, 1978; Benn *et al.*, 1985).

4. REDUCTION OF THE DIRAC EQUATION

Let us now address the problem of reduction of the Dirac equation from the four-dimensional space to its two-dimensional submanifold. We shall discuss this point within the framework of the approach of Rashevsky (1955), which establishes a natural connection between spinors and differential forms.

As mentioned above, the spinor space is a minimal left ideal of the Clifford algebra, and it is isomorphic to the complex Clifford algebra $C_{1,1}$ of an auxiliary two-dimensional space (spanned by the vectors e_0 and e_1 in the case under consideration).

So, let us consider nonhomogeneous exterior forms

$$\psi = \sum_{k=0}^{2} \frac{1}{k!}\,\varphi_{a_1 \cdots a_k} e^{a_1} \wedge \cdots \wedge e^{a_k} \tag{4.1}$$

which in terms of Rashevsky (1955) will be called spinors. The set of such forms will be called the spinor space hereafter. An orthonormal 1-form basis in the Minkowski space M^4 is denoted $\{e_A, A = 0, 1, 2, 3\}$. Its choice is not connected with the introduction of some coordinates in M^4, and hence an element (4.1) is [like the element (2.1)] invariant, independent of spacetime coordinates.

We now describe the action of the orthonormal group of M^4 on the spinor space. Following Rashevsky (1955), we notice that each 1-form $e_A, A = 0, 1, 2, 3$, determines a linear operator E_A which acts on the elements (4.1) according to the rules

$$
\begin{aligned}
e_{i,i} = 0, 1: \quad & E_i\psi = e_i \vee \psi \\
e_2: \quad & E_2\psi = i\alpha(\psi) \vee e_1 \\
e_2: \quad & E_3\psi = \alpha(\psi) \vee e_0
\end{aligned}
\tag{4.2}
$$

where α is the main involution (3.16).

These operators satisfy

$$E_A E_B + E_B E_A = 2\eta_{AB}$$

and they are evidently in correspondence with the above operators $\{\hat{E}_A\}$, (3.13), which act on the space of algebraic spinors. Hence, just as before, we can define the action of the orthogonal group on (4.1) by (substituting $e_A \to E_A$) mapping, say, the infinitesimal rotation (3.15), $Y = 1 + \frac{1}{4}\omega_{\alpha\beta}e^\alpha e^\beta$, to the operator $Y = 1 + \frac{1}{4}\omega_{\alpha\beta}E^\alpha E^\beta$. In this way we naturally introduce the action of the spinor group (Pin and Spin) on the spinor space (4.1).

Now we are in the position to write the Dirac equation,

$$iE^A\partial_{e_A}\psi - m\psi = 0 \tag{4.3}$$

Here $\{\partial_{e_A}\}$ is the vector basis dual to $\{e_A\}$.

Let us fix in the Minkowski space-time M^4 an orthogonal coordinate system (x^0, x^1, x^2, x^3) and consider the two-dimensional plane $M^2 = \{(x^0, x^1, x^2, x^3) \in M^4 | x^2 = x^3 = 0\}$. Then we can put $e^0 = dx^0, e^1 = dx^1$ in (4.1)–(4.3) (the vectors $\partial/\partial x^0$ and $\partial/\partial x^1$ are tangent to M^2). We are interested now in the restriction of the Dirac equation (4.3) on the plane M^2. For this purpose we choose for the spinor field ψ components in the neighborhood of M^2 the natural *ansatz*

$$\psi^i(x^0, \ldots, x^3) = \psi^i(x^0, x^1) + C^i_{\mu\nu}(x^0, x^1)x^\mu x^\nu + \cdots$$

That is, one has $\partial_\mu\psi^i|_{x\in M^2} = 0$, $\mu = 2, 3$, in this case.

Hence, on the plane M^2 the Dirac equation (4.3) reduces to

$$\left(iE^a\frac{\partial}{\partial x^a} - m\right)\psi = 0$$

or, using (4.2),

$$\left(i\,dx^a \vee \frac{\partial}{\partial x^a} - m\right)\psi = 0$$

This is equivalent to the ILK equation on M^2,

$$\{i(d-\delta) - m\}\psi = 0$$

It would be interesting to generalize such a construction to the case of an arbitrary (not necessarily flat) two-dimensional surface $R^2 \subset R^4$. In order to study this problem we need the description of the spinor space and the relevant Dirac equation on a Riemannian manifold R^4.

Let $\{e_A, A = 0, 1, 2, 3\}$ be a local orthonormal basis on R^4 with the metric $g(e_A, e_B) = \eta_{AB}$. As above, we define the spinor space in R^4 as the set of forms (4.1). Assuming R^4 is supplied with the zero-torsion metric connection, one introduces covariant derivatives,

$$\begin{aligned} \nabla e^A &= de^A + \omega^A{}_B \wedge e^B = 0 \\ \nabla_\alpha \varphi_A &= \partial_\alpha \varphi_A - \omega^B{}_{A,\alpha} \varphi_B \end{aligned} \tag{4.4}$$

where $\omega^A{}_B = \omega^A{}_{B,\alpha}\, dx^\alpha$ is the Lorentz connection 1-form.

It is useful to notice that the spinor ψ is in fact an element (2.1) where all components which involve the indices (2) and (3) are zero. Hence for the covariant derivative of the form ψ, considered as the geometrical object in R^4, we have

$$\begin{aligned} \nabla_\alpha \psi = {} & (\partial_\alpha \varphi + \partial_\alpha \varphi_a e^a + \tfrac{1}{2}\partial_\alpha \varphi_{ab} e^a \wedge e^b) \\ & - (\omega^c{}_{a,\alpha} \varphi_c e^a + \omega^c{}_{a,\alpha} \varphi_{cb} e^a \wedge e^b + \omega^c{}_{b,\alpha} \varphi_{ac} e^a \wedge e^b \end{aligned} \tag{4.5}$$

where the indices a, b, c run from 0 to 1. One easily sees that the two last terms in (4.5) are zero, hence

$$\nabla_\alpha \psi = \partial_\alpha \psi - \omega^a{}_{b,\alpha} \varphi_a e^b \tag{4.6}$$

and $\partial_\alpha \psi$ denotes the first three terms in (4.5).

Notice that $\nabla_\alpha \psi$ also belongs to the spinor space, i.e., it has the form (4.1). So one can generalize the Dirac equation (4.3) to the Riemannian case as follows:

$$iE^A h^\alpha_A \nabla_\alpha \psi - m\psi = 0 \tag{4.7}$$

where E^A are the Dirac operators (4.2) and h^α_A is the orthonormal basis (tetrad) field, $e^A = h^A_\alpha\, dx^\alpha$. Such a curved-space Dirac equation is different from the standard one (Fock and Ivanenko, 1929; Penrose and Rindler, 1986).

Let us now consider the curved two-dimensional surface R^2 in R^4, which in local coordinates is defined by $R^2 = \{(x^0, x^1, x^2, x^3) \in R^4|$

$x^2 = x^3 = 0\}$. We assume that e^0 and e^1 are tangent to R^2, i.e., $e^0 = h_i^0\, dx^i$, $e^1 = h_i^1\, dx^i$. To study the restriction of (4.7) on R^2 it is convenient to choose the *ansatz* for the components of spinor field ψ in the neighborhood of R^2 in the form

$$\begin{aligned}\varphi(x^0, \ldots, x^3) &= \varphi(x^0, x^1) + a_{\mu\nu}(x^0, x^1)x^\mu x^\nu + \cdots \\ \varphi_{01}(x^0, \ldots, x^3) &= \varphi_{01}(x^0, x^1) + b_{\mu\nu}(x^0, x^1)x^\mu x^\nu + \cdots \\ \varphi_a(x^0, \ldots, x^3) &= \varphi_a(x^0, x^1) + \omega^b{}_{a,\mu}(x^0, x^1)\varphi_b(x^0, x^1)x^\mu + \cdots\end{aligned} \tag{4.8}$$

where the dots denote higher orders in x^2 and x^3. In this case (4.4)–(4.6) yield

$$\nabla_\mu \psi|_{x \in R^2} = 0, \qquad \mu = 2, 3 \tag{4.9}$$

Hence, (4.7) induces on R^2 the equation

$$(iE^a h_a^i \nabla_i - m)\psi = 0 \tag{4.10}$$

or

$$(ie^a \vee \nabla_a - m)\psi = 0$$

which is equivalent to the ILK equation on R^2.

As a remark, let us point out that the spinor space (4.1) may be connected with a given pair of vectors $\{e^a\}$, but the reduction can be constructed with respect to a surface R^2 with tangent vectors $\{\bar{e}^a\}$ which are different from $\{e^a\}$. In this case the result will be the same in view of the invariance of (4.7) under the orthogonal rotations of the basis $e^a \to Ve^aV^{-1}$.

For comparison one can consider the standard Dirac equation in the Riemannian space (Penrose and Rindler, 1986). Defining the spinor connection as usual,

$$\overset{S}{\nabla}_\alpha \psi = \partial_\alpha \psi + \Gamma_\alpha \psi \tag{4.11}$$

with $\Gamma_\alpha = \frac{1}{4}\omega_{AB,\alpha} E^A E^B$, one finds that the Dirac equation reads

$$iE^A h_A^\alpha \overset{S}{\nabla}_\alpha \psi - m\psi = 0 \tag{4.12}$$

In the neighborhood of R^2 the spinor components ψ^i are assumed to be as follows:

$$\psi^i(x^0, \ldots, x^3) = \psi^i(x^0, x^1) - \Gamma_\mu{}^i{}_j(x^0, x^1)\psi^j(x^0, x^1)x^\mu + \cdots \tag{4.13}$$

where $\Gamma_\mu{}^i{}_j$ is the matrix of the spinor connection. Hence

$$\overset{S}{\nabla}_\mu \psi|_{x \in R^2} = 0, \qquad \mu = 2, 3$$

and the reduction of (4.12) on R^2 yields

$$(iE^a h_a^i \overset{S}{\nabla}_i - m)\psi = 0 \tag{4.14}$$

For the connection (4.6) we have evidently

$$\nabla_i \psi = \partial_i \psi + S_i \vee \psi - \psi \vee S_i \tag{4.15}$$

where $S_i = \frac{1}{4}\omega_{ab}, ie^a \wedge e^b$.

One can see that equation (4.14) contains terms which depend on $\omega_{nm,i}$ and $\omega_{an,i}$, $n, m = 2, 3$, and the latter are determined not only by the geometry of R^2, but also by the embedding of R^2 into R^4. Clearly, when the four-dimensional space has the direct product structure $R^4 = R^2 \otimes \tilde{R}^2$, these terms are zero, and the induced equation on R^2 reads

$$i(d - \delta)\psi + ie^a \vee \psi \vee S_a - m\psi = 0 \tag{4.16}$$

This coincides with the modified ILK equation suggested by Benn and Tucker (1985; Bullinaria, 1986), which admits decomposition into the algebraic spinors.

Finally, one more remark is in order. As one knows (Graf, 1978; Benn *et al.*, 1985; Bullinaria, 1986), in general the field (2.1) cannot be decomposed on algebraic spinors in such a way that the ILK equation (2.3) splits into several independent Dirac equations, because an arbitrary manifold does not admit covariantly constant idempotent fields, i.e., in general $\nabla_\mu(\phi \vee P_i) \notin S_i$. Hence, algebraic spinors do not always exist.

However, the spinor (4.1) considered above is not an algebraic spinor, although relevant spinor spaces are isomorphic in the Minkowski space. As we noticed, the natural covariant derivative $\nabla_\mu \psi$ is again the element of the form (4.1); the standard covariant derivative $\overset{S}{\nabla}_\mu$ also has this property, i.e., $\overset{S}{\nabla}_\mu \psi \in S$.

We have thus described a new nonequivalent method of introducing spinors on a Riemannian space. Spinors are then analogous to the usual tensor objects.

5. CONJUGATED SPINORS

One needs the conjugation operation in the spinor space. Let us recall that the metric signature in M^4 is $(+1, -1, -1, -1)$.

For any two nonhomogeneous exterior forms

$$\begin{aligned} \psi &= f + f_0 e^0 + f_1 e^1 + f_{01} e^0 e^1 \\ \phi &= \varphi + \varphi_0 e^0 + \varphi_1 e^1 + \varphi_{01} e^0 e^1 \end{aligned} \tag{5.1}$$

the scalar product is defined (Bullinaria, 1986)

$$\langle \psi | \phi \rangle = \int (f\varphi + f_0\varphi_0 - f_1\varphi_1 + f_{01}\varphi_{01}) * 1 \tag{5.2}$$

which has the properties

$$\langle \psi, \phi \rangle = \langle \phi, \psi \rangle$$
$$\langle \psi, \phi \rangle = \langle \alpha(\phi), \alpha(\psi) \rangle$$
$$\langle \psi, \phi \rangle = \langle \beta(\phi), \beta(\psi) \rangle$$

One easily sees that

$$\langle \psi, \phi \rangle = (\beta(\psi), \phi) \tag{5.3}$$

where $(\cdot\,,\cdot)$ is the natural scalar product of p-forms,

$$(\psi^{(p)}, \phi^{(p)}) = \int * \psi^{(p)} \wedge \phi^{(p)} \tag{5.4}$$

Let us define the Hermitian conjugated spinor

$$\psi^+ = e_0\beta(\psi^*)e_0 = \bar{f} + \bar{f}_0 e_0 - \bar{f}_1 e^1 + \bar{f}_{01} e^0 e^1 \tag{5.5}$$

and the Hermitian product

$$\langle \psi^+, \phi \rangle = \int (\bar{f}\varphi + \bar{f}_0\varphi_0 + \bar{f}_1\varphi_1 + \bar{f}_{01}\varphi_{01}) * 1 \tag{5.6}$$

As usual, the conjugated operator is given by

$$\langle \psi^+, \hat{A}\phi \rangle^* = \langle \phi^+, \hat{A}^+\psi \rangle \tag{5.7}$$

Then for the operators (4.2) one finds

$$\begin{aligned} E_0^+ &= E_0; \qquad & E_1^+ &= -E_1 \\ E_2^+ &= -E_2; \qquad & E_3^+ &= -E_3 \end{aligned} \tag{5.8}$$

The Dirac conjugated spinor $\bar{\psi} = \psi^+\bar{E}$ must satisfy

$$\langle \bar{\psi}, E_\alpha \phi \rangle^* = \langle \bar{\phi}, E_\alpha \psi \rangle$$

from which $E_\alpha = EE_\alpha^+ E$ under the condition of hermiticity of E. The latter can be chosen as follows, $E = E_0$:

$$\bar{\psi} = \psi^+\bar{E}_0 = e_0\beta(\psi^*) \tag{5.9}$$

Then the conjugated spinor satisfies the equation

$$i\partial_\alpha \bar{\psi}\bar{E}^\alpha + m\bar{\psi} = 0 \tag{5.10}$$

and it is transformed with the help of the inverse matrix under the spinor Lorentz transformations in spinor space. For example, in the infinitesimal case

$$\begin{aligned} \psi' &= \hat{S}\psi = (1 + \tfrac{1}{2}\omega_{\alpha\beta}E^\alpha E^\beta)\psi \\ \bar{\psi}' &= \overline{S\psi} = \bar{\psi}(1 - \tfrac{1}{2}\omega_{\alpha\beta}\tilde{E}^\alpha\tilde{E}^\beta) = \bar{\psi}\hat{S}^{-1} \end{aligned} \tag{5.11}$$

Now one can easily determine the Dirac action

$$I = \frac{i}{2}\langle\bar{\psi}, E^\alpha\partial_\alpha\psi\rangle - \frac{i}{2}\langle\partial_\alpha\bar{\psi}\tilde{E}^\alpha, \psi\rangle - m\langle\bar{\psi}, \psi\rangle \tag{5.12}$$

Restriction to the two-dimensional plane M^2 reduces (5.12) to

$$I = i\int * \beta(\bar{\psi}) \wedge (d - \delta)\psi - m\int * \beta(\bar{\psi}) \wedge \psi \tag{5.13}$$

Thus, the reduction induces on the 2D surface not only the equation of motion, but the conjugation recipe as well. In the present case (5.9) yields $\bar{\psi} = \psi^+ \vee e_0$.

Let us note that the conjugation law depends on the signature of the 4-dimensional space. For example, suppose we start from a different one, the metric of which has the signature $(+1, -1, +1, -1)$. Then

$$\begin{aligned} E_0\psi &= e_0 \vee \psi, \qquad & E_2\psi &= \psi \vee e_0 \\ E_1\psi &= e_1 \vee \psi, \qquad & E_3\psi &= i\alpha(\psi) \vee e_1 \end{aligned} \tag{5.14}$$

and for the operators which are conjugate with respect to the scalar product (5.6) one finds

$$\begin{aligned} E_0^+ &= E_0, \qquad & E_1^+ &= -E_1 \\ E_2^+ &= E_2, \qquad & E_3^+ &= -E_3 \end{aligned} \tag{5.15}$$

The condition $E_\alpha = EE_2^+E$ then yields that $E = E_0E_2$ and hence the Dirac conjugated spinor reads now

$$\bar{\psi} = \psi^+\tilde{E} = e_0\psi^+e_0 = \beta(\psi^*) \tag{5.16}$$

The observed difference is rather essential: while in the first case the charge $\langle\bar{\psi}, e_0\psi\rangle$ is a positive-definite quantity, it is indefinite for an anomalous signature. We discuss this in the next section.

6. REMARKS ABOUT THE ILK EQUATION IN M^4

The relation between the Dirac and ILK equations established above in flat space is in fact quite universal. Not entering into the details of describing the reduction to a nonflat surface, one may say that the ILK equation in M^n is obtained from the restriction of the Dirac equation in M^{2n} on the subspace $M^n \subset M^{2n}$. In particular, the ILK equation in M^4 can be considered as the reduction of the Dirac equation from the 8-dimensional space.

Let us discuss such a construction. Let φ be the Dirac spinor in M^8 with the metric of signature $(1, 7)$. It satisfies the Dirac equation

$$\sum_{a=0}^{7} i\Gamma^a \partial_a \varphi - m\varphi = 0 \tag{6.1}$$

where the 8-dimensional Γ-matrices

$$\Gamma^a \Gamma^b + \Gamma^b \Gamma^a = 2\eta^{ab}$$

can be chosen in the form

$$\begin{aligned} \Gamma^\mu &= 1 \otimes \gamma^\mu, \qquad \mu = 0, \ldots, 3 \\ \Gamma^4 &= \gamma^0 \otimes \gamma^5 \\ \Gamma^{4+j} &= i\gamma^j \otimes \gamma^5, \qquad j = 1, 2, 3 \end{aligned} \tag{6.2}$$

One easily verifies that

$$\Gamma_0 \Gamma_a^+ \Gamma_0 = \Gamma_a \tag{6.3}$$

where $\Gamma_0 = 1 \otimes \gamma_0$.

Hence the conjugated spinor is defined as

$$\bar{\varphi} = \varphi^+ \Gamma_0 \tag{6.4}$$

and the Dirac Lagrangian reads

$$L = i\bar{\varphi}\Gamma^a \partial_a \varphi - m\bar{\varphi}\varphi \tag{6.5}$$

Reduction to $M^4 \subset M^8$ gives the equation

$$\sum_{\mu=0}^{3} i\Gamma^\mu \partial_\mu \varphi - m\varphi = 0 \tag{6.6}$$

which is evidently the ILK equation written in the form (2.15).

Alternatively, (6.6) may be easily rewritten in the form (2.12) for the 4×4 matrix field ψ:

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

with the Lagrangian similar to (2.14):

$$L = i\,\mathrm{Tr}(\bar{\psi}\gamma^\mu \partial_\mu \psi) - m\,\mathrm{Tr}(\bar{\psi}\psi) \tag{6.7}$$

where, however, now $\bar{\psi} = \psi^+ \gamma_0$.

The Lagrangian (6.7), unlike (2.14), describes the ILK equation with the compact internal symmetry group $SU(4)$. Now we can explain the origin of the noncompact symmetry in (2.14): it arises from the reduction from a space with anomalous signature (4, 4).

Indeed, in such a case the Γ matrices read

$$\begin{aligned} \Gamma^{\mu} &= 1 \otimes \gamma^{\mu}, \qquad \mu = 0, \ldots, 3 \\ \Gamma^{4+\mu} &= \gamma^{\mu} \otimes \gamma^{5} \end{aligned} \tag{6.8}$$

and consequently the Dirac conjugation matrix C is as follows:

$$C\Gamma_a^{+}C = \Gamma_a, \qquad C = \gamma_0 \otimes \gamma_0 \tag{6.9}$$

Then the 8-dimensional conjugated spinor is defined by

$$\bar{\varphi} = \varphi^{+}C \tag{6.10}$$

and hence on M^4 the induced Lagrangian takes the form (2.14),

$$L = i\,\mathrm{Tr}(\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi) - m\,\mathrm{Tr}(\bar{\psi}\psi) \tag{6.11}$$

where $\bar{\psi} = \gamma_0 \psi^{+} \gamma_0$.

In summary, the conjugation law for the ILK field and consequently the internal ("right") symmetry group of the ILK theory are determined by the signature of the metric of the 8-dimensional space M^8. The noncompact symmetry group arises when M^8 has more than one timelike coordinate. This is apparently an unphysical situation, which probably sheds light on the difficulties of quantization of the ILK field discovered earlier (Benn and Tucker, 1983; Ivanenko *et al.*, 1985; Ivanenko and Obukhov, 1985; Leonovich, 1983; Satikov and Stragev, 1987).

7. CONCLUSION

We have shown that the nonhomogeneous differential forms suggest a new framework in which one can construct a natural theory of spinors. The spinor turns out to be a "two-dimensional" object, in the sense that it is related to a two-dimensional space (plane, determined by a pair of tangent vectors). Hence it is not very unexpected to find that the reduction of the Dirac equation to this two-space induces the ILK equation on it. This is a general property which naturally relates the ILK equation on M^n with the Dirac equation in M^{2n}. As a by-product we have revealed the origin of the difficulties in the quantization of the ILK field in M^n: these arise when M^{2n} possesses several timelike coordinates. In summary, an interesting hierarchy of the ILK–Dirac relations can be established: the ILK equation in M^{2^n} is equivalent to $2^{2^{n}-1}$ Dirac equations each of which under the reduction to $M^{2^{n}-1} \subset M^{2^n}$ yields an ILK equation, etc.

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